An alternative definition for a fractal set in a D-dimensional space

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In this article fractal set has been interpreted as a collection of fixed points of some functions. Hence the set remains invariant by the functions that defines the fractal set.

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Let us first define a mathematical notation as follows:

$$\bigcirc_{j=1}^{k} f_j = f_1 \circ f_2 \circ \cdots \circ f_k = f_1(\,f_2(\,\cdots\,(\,f_k(\,)\,)\cdots)\,)$$

Now,
Let us consider a D-dimensional space $\mathbb{S}^D$. [D being a positive integer.]
Let a point $(x_1, x_2, ..., x_D)$ in the space be **p**.

Let 'f_i' be a function such that $f_i: \mathbb{S}^D \to \mathbb{S}^D$

Let $f_1, f_2, ... f_n$ be *n* given functions.
And let, $f_0(\mathbf{p}) = \mathbf{p}$

We define a set of fixed points of a function '*g*' as $F_g = \{\mathbf{p}: g(\mathbf{p}) = \mathbf{p}\}$

Hence, the set F_k

$$F_k = \{\mathbf{p} : \;\; \mathbf{p} = (\bigcirc_{j=1}^{k} f_{i_j})(\mathbf{p}) \;, \text{ where } i_j \text{ is an integer between 0 and n for any j.} \}$$

is the set of fixed points of sequence of k functions $f_{i_1}, f_{i_2}, ... f_{i_k}$.
i.e. it is the set of fixed points of the function $(\,f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k})$

Now, a fractal set is defined as the union of all possible F_k s for all possible values of $i_1, i_2, ... i_k$, as k tends to infinity.

Hence,
Fractal set =

$$F = \operatorname*{Lt}_{k \to \infty} \left[\bigcup_{i_1=0}^{n} \bigcup_{i_2=0}^{n} \cdots \bigcup_{i_k=0}^{n} \{\mathbf{p} : \;\; \mathbf{p} = (\bigcirc_{j=1}^{k} f_{i_j})(\mathbf{p}) \} \right]$$

The functions $f_1, f_2, ... f_n$ are called the 'transformations' that define the fractal set.

From this definition we note:
$f_i(F) = F$, for i = 0, 1, 2, ... n.
That is, a fractal set (F) is invariant under any of the 'transformations' that define the fractal set.

References:

1. Fractal Geometry by Mandelbrot.