Geometric and Topological Techniques in Graph Search-based Robot Planning

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1 Extended Abstract

Search-based techniques have been widely used in robot path planning for finding optimal trajectories in configuration spaces. They have the advantages of being complete, optimal (up to the metric induced by the discretization) and efficient (in low dimensional problems), even in complex environments. Continuous techniques, on the other hand, lending concepts from differential and algebraic geometry and topology, have the ability of exploit specific structures in the original configuration space and help in solving a host of different problems that rarely come under the scope of graph-search based techniques. The main objective of this thesis is to propose certain ideas and methods that will let us bring these two separate techniques under one umbrella.

The first contribution of the thesis lies in the characterization of the topology of the configuration space and the solution using applications of discrete algebraic geometry and discrete algebraic topology. Search-based techniques ignore the continuous properties of the underlying configuration space and the solution trajectories. Thus, we define differential forms whose integral reveal metric and topological information about the solution path. For example, the volume 1-form (*i.e.* length) integrated along the edges of the search graph yields the total length of the path. Similarly other appropriately defined 1-forms allow us to establish equivalence classes of trajectories (*e.g.*, homotopy classes) and use it to guide the search. We show how to find trajectories that are constrained to lie in specified homotopy classes or that avoid other specified homotopy classes.

The second contribution of the thesis centers around the determination of optimal trajectories for specified metrics. Trajectories found by discrete graph representations and searches suffer from sub-optimality induced by the discretization. However there are certain metric spaces (a trivial example being the Euclidean metric) in which we can conveniently construct certain types of graphs (*e.g.,* the visibility graph) in which the optimal trajectory on the graph is also the optimal trajectory with respect to the metric. Our goal is to establish the existence of such special metric spaces and define conditions under which we can transform a given non-Euclidean metric space into such special metric spaces.

Our third contribution is to use search techniques to partition the configuration space to facilitate multi-robot tasks. We show how lending concepts from dual simplices discrete graph representations can be used to efficiently compute a weighted volume of any subset of the configuration space and hence develop a search technique for creating partitions of the configuration space. In particular, we illustrate the computation of Voronoi partitions with applications to multi-robot exploration and coverage of unknown or partially known non-convex environments.

Finally, we address the curse of dimensionality that is inherent in path planning for multi-robot systems. One of the main drawbacks of graph search algorithms is that with increase in the dimensionality of the configuration space, the number of nodes and edges in the graph increase exponentially. This poses a major challenge for finding optimal paths in high dimensional configuration spaces using graph search techniques. While gradient descent approaches scale much better with the dimensionality of the configuration space, these methods suffer from local minima, especially in non-convex environments. However, multirobot problems endow a special product structure to the configuration space allowing us to decouple robot directions and parallelize the search. Such decompositions can let us a combination of graph search methods and gradient descent algorithms in complementary directions. We demonstrate how such decompositions are particularly suitable for multi-robot path planning problems with communication constraints.

2 Brief Literature Survey

Robot path planning is probably one of the most extensively studied problems in robotics [21]. There has been extensive research on path planning with a variety of constraints such as communication constraints [1], dynamics and environment constraints [23] and time constraints [17].

Despite being mostly an uncharted research area in robotics, constraints imposed by equivalence relations like homotopy often appear in path planning problems. For example, in multi-agent planning problems [36, 20], the trajectories often need to satisfy certain proximity or resource constraints or constraints arising due to tasks allocated to agents, which translates into restricting the solution trajectories to certain homotopy classes. In exploration and mapping problems [9], agents often need to plan trajectories based on their mission or part of the environment they are assigned for mapping or exploration, and hence restrict their trajectories to certain homotopy classes. Motion planning with homotopy class constraints have been studied in the past using geometric approaches [16, 18] and probabilistic road-map construction [30] techniques. Such techniques suffer from complexity of representation of homotopy classes and are not immediately integrable with standard graph search techniques. While comparing trajectories in different homotopy classes and finding the different homotopy classes in an environment is possible using such techniques, optimal path planning with homotopy class constraints is not achievable in an efficient way without a invariant that is not additive function of trajectories. Moreover, most of such methods in robotics literature are primarily restricted to 2-dimensional configuration spaces. We hence propose an additive invariant for a homotopy-like equivalence classes of trajectories which can be incorporated into graph-search based algorithms for finding least-cost paths in Euclidean configuration spaces independent of the geometry of discretization, cost function or search algorithm. While technically the equivalence relation that we use is different from the exact notion of homotopy [6], it serves as a good practical tool for robotics planning problems where homotopy constraints arise naturally. The study of equivalence classes of sub-manifolds of high dimensional complex and real manifolds is not new. The *generalized Residue Theorem* in high dimensional complex manifolds [15] and Clifford algebra [12] are highly developed along similar lines. Homology theory [19] has been highly developed for identification of *homology classes* of sub-manifolds in arbitrary topological spaces. Another recent development in the study of equivalence relations between manifolds is cobordism theory [24, 25]. Cobordism is a much broader equivalence relation, and forms the basis for surgery theory. In the mathematical development of our method [6] we extensively use some of the concepts from cobordism theory.

For a given arbitrary metric space, one can attempt to immerse or embed it in an Euclidean metric space. However, in general, if the dimensionality of the given metric space is same as the Euclidean space in which we are trying to embed it, one may not be able to find an isometric embedding. In the 2-dimensional version of the problem, interesting exceptions are parabolic Riemann surfaces, which can be transformed into the flat Euclidean metric using conformal mappings [31], thus can be isometrically embedded in $\mathbb{R}^2$. The flatness of Euclidean metric space lets us compute geodesics and check their intersections with significant ease, hence exploit notions like visibility for solving many motion planning problems. Otherwise, finding geodesics between two points in arbitrary metric spaces can be an computationally expensive, especially for utilizing it for visibility-based techniques. Results obtained in the Euclidean space can then be transformed back to the original metric space of the problem. An interesting and significant result of similar interest is the C^∞ Embedding theorem due to J. F. Nash [26] which states that any n-dimensional metric space can be isometrically embedded in a Euclidean space of dimension no greater than $n(3n + 11)/2$.

The problem of path planning also forms an important sub-set of the problem of exploration and coverage of environments. A common approach toward exploration is frontier-based exploration where control directions seek to minimize entropy or uncertainty in the robot pose or map [34]. In [33], the authors propose an exploration strategy with feedback control laws that maximize information gain by considering uncertainty in both the robot pose and map. A multi-robot exploration strategy is presented in [10, 32], where the robots coordinate to determine targets best served by each robot that maximize the information gain for the team of robots. A common coverage control approach is through the definition of feedback control laws defined with respect to the centroids of Voronoi cells resulting from the Voronoi tessellation of an environment. In [11], the authors propose gradient descent-based individual robot control laws that guarantee optimal coverage of a convex environment given a density function which represents the desired coverage distribution. By modeling the information and entropy of the environment as a metric of the space we utilize graph search-based approaches for achieving both exploration and coverage.

In context of multi-robot planning with constraints, distributed optimization is an indispensable and well-investigated approach for tackling the dimensionality of the configuration space. Task allocation for multiple robots [14] using auction-based solutions [13], separable optimization problems [3] with linear constraints using techniques based on dual decomposition [29, 3], and solution to similar problems using augmented Lagrangian type methods [2, 27] have been used for solving simple classes of large constrained optimization problems using iterative methods. However such methods are limited to problems with linear constraints, rely on convexity of cost functions, or provide no guarantee on optimality due to auction-based approaches. We investigate a distributed implementation of a separable optimization problem with *non-linear* constraints arising from coupling between pairs of robots.

3 Work done so far

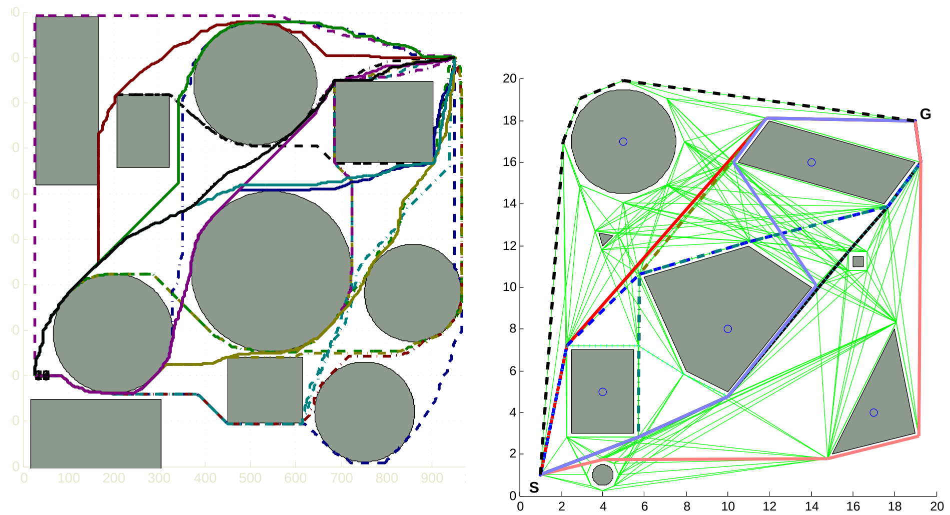
Below is a brief review, along with references to relevant papers, of the work done so far.

3.1 Planning with Homotopy class constraints

The motivation of this sub-problem arises from goal-directed path planning with homotopy class constraints for robots in Euclidean configuration spaces. There are many applications in motion planning where it is important to distinguish between and consider the different *homotopy classes* of trajectories. Two trajectories in a configuration space are homotopic if one trajectory can be continuously deformed into another without passing through an obstacle, and a homotopy class is a collection of homotopic trajectories. In this section we consider the problem of robot exploration and planning in Euclidean configuration spaces to identify and classify different homotopy classes and plan trajectories constrained to certain homotopy classes or avoiding some others. We desire the methods we develop be independent of the discretization scheme, the cost function or geometry of the environment, so that it can be .

3.1.1 The problem in 2 dimensions

In the first part of this work [5] we solve this problem for two-dimensional, static environments using the Cauchy Integral Theorem in concert with incremental graph search techniques (Figure 1). The robot workspace is mapped to the complex plane and obstacles are modeled as homotopy equivalents of poles of certain complex analytic functions (*Obstacle Marker Function*) defined on this plane. The Residue Theorem then allows an efficient way of representing homotopy classes of trajectories using the line integral of the Obstacle Marker Function over the trajectory in a complex plane. Using the proposed representation, we have shown that homotopy class constraints can be directly weaved



(a) Paths in 20 different homotopy classes in a 1000 × 1000 uniformly discretized environments

(b) Exploring homotopy classes using a Visibility Graph

Figure 1: Least cost trajectories in different homotopy classes in a 2-dimensional configuration space with obstacles.

with graph search techniques for determining optimal path constrained to certain homotopy classes or forbidden from others. We have proved the optimality of the method and have experimentally demonstrated its efficiency, versatility and several applications.

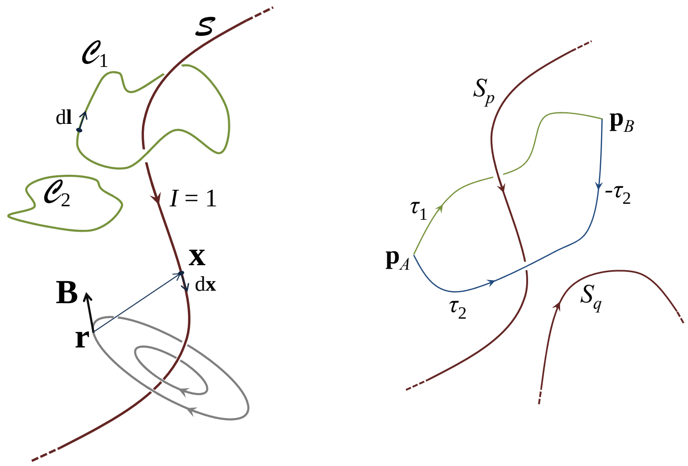
3.1.2 The problem in 3 dimensions

An immediate extension of the above described problem is to plan in 3-dimensional configuration spaces with similar constraints [7]. In a 3-dimensional Euclidean space the notion of homotopy classes can only be induced by obstacles with *genus* one or more, or by obstacles stretching to infinity in two directions. Upon performing some construction on the obstacles, we can create certain 1-dimensional closed loops inside the obstacles that we call *skeletons*. We model those as current-carrying conductors or wires in a 3-dimensional Euclidean space. This construction immediately lets us compute a *virtual magnetic field* induced due to the current in each skeleton using the Biot-Savart's law from electromagnetism. Thus, for any given robot trajectory, we can compute a line integral of the virtual magnetic fields using the Ampere's law, giving us a homotopy-like class invariant (*h-signature*) for the trajectory. Once again, being an additive function, *h*-signatures can be efficiently incorporated into graph search techniques for,

i. exploring different homotopy classes in an environment (Figure 3(a)), and

ii. determining optimal path constrained to certain homotopy classes or forbidden from others (Figure 3(b)).

The configuration spaces we use to demonstrate the proposed method are those of static 3-dimensional obstacles ($X - Y - Z$ configuration space) and dynamic 2-dimensional obstacles ($X - Y - Time$ space).



(a) Magnetic field due to current in a conductor $\mathcal{S}$, and its integration along a closed loop $\mathcal{C}_i$.

(b) Two trajectories, τ_1 and τ_2, connecting the same points form a closed loop.

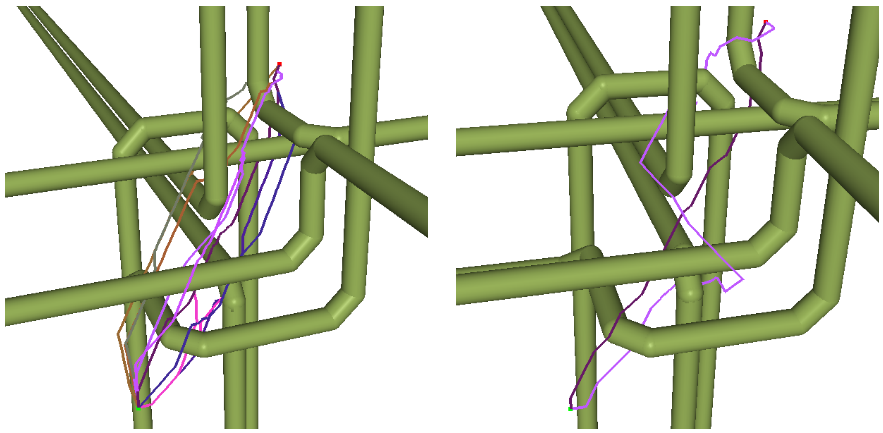
Figure 2: The Biot-Savart's law and the Ampere's law can be used for identifying homotopy classes of trajectories in 3-dimensional configuration spaces.

Strictly speaking, this equivalence relation under consideration is *homology* rather than *homotopy*. However in our analysis we used certain concepts from *cobordism theory* rather than *homology* to formally define the equivalence relation. Since trajectories in different homology classes are guaranteed to be in different homotopy classes as well, the fact that the invariant we developed is a homology invariant does not effect the problems where we attempt to explore multiple homotopy classes using the method described above. However, given a set of *h-signature*s, planning with homotopy class constraints may not result in the desired result under certain pathological cases since what we really imposing are the homology class constraints. However, in most practical cases, the fact that trajectories of robots are homeomorphic (homeomorphic to $[0, 1]$), and with the assumption that obstacles in real 3D environments rarely form links or braids, planning using the method described above with given *h-signature*s constraints, give us plans with the desired homotopy class constraints as well.

3.1.3 Generalization and extension to higher dimensions

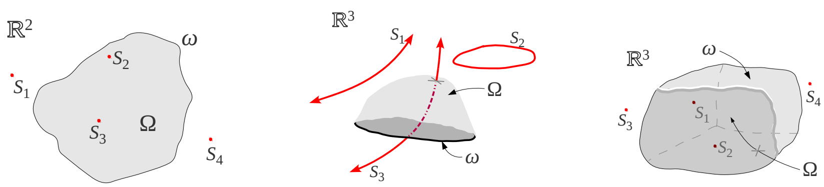
From the previous discussions, it is not difficult to see an underlying connection between the Residue theorem from complex analysis and the Biot-Savart law and Ampere's law from electromagnetism. Both triy to prescribe an equivalence relation for 1-dimensional manifolds (curves) in an Euclidean space punctured by some other manifolds. In fact, it is also easy to find a similar connection to Gauss divergence theorem, which prescribes equivalence relation for surfaces, when the Euclidean 3-space is punctured by certain points (Figure 4). Upon further investigation it can be concluded that the equivalence relation underlying these laws is same as the equivalence of *homology* in punctured Euclidean spaces. However, as mentioned before, in our investigation, we used concepts from cobordism theory to define the equivalence relation under study.

Thus, in the final part of this problem we seek to investigate this equivalence relation, obtain



(a) Exploring 10 distinct homotopy classes. (b) Plan in the *complementary homotopy class* of the least cost path.

Figure 3: An environment with 7 unbounded pipes.



(a) $D = 2, N = 2$ (b) $D = 3, N = 2$ (c) $D = 3, N = 3$

Figure 4: The Cauchy Residue theorem can be applied to (a), Ampere's law to (b), and Gauss Divergence theorem to (c), for defining an equivalence relation for the manifolds ω. There is an underlying connection between these different cases.

a unified theory for it, and generalize it to higher dimensions [6]. In particular, we consider the D-dimensional Euclidean space, $\mathbb{R}^D$, with certain $(D-N)$-dimensional compact, closed and orientable sub-manifolds (which we call *singularity manifolds* and represent by $\widetilde{\mathcal{S}}$) removed from it. We define and investigate the problem of finding a homotopy-like class invariant (χ-homotopy) for certain $(N-1)$-dimensional compact, closed and orientable sub-manifolds (which we call *candidate manifolds* and represent by ω) of $\mathbb{R}^D \setminus \widetilde{\mathcal{S}}$. We determine a differential $(N-1)$-form, $\psi_{\widetilde{\mathcal{S}}}$, such that $\chi_{\widetilde{\mathcal{S}}}(\omega) = \int_\omega \psi_{\widetilde{\mathcal{S}}}$ is a class invariant for such candidate manifolds. We show that the formula agrees with formulae from Cauchy integral theorem and Residue theorem of complex analysis (when $D = 2, N = 2$), Biot-Savart law and Ampere's law of theory of electromagnetism (when $D = 3, N = 2$), and the Gauss divergence theorem (when $D = 3, N = 3$), and discover that the underlying equivalence relation suggested by each of these well-known theorems is the χ-homotopy of sub-manifolds of these low dimensional punctured Euclidean spaces. We have described numerical techniques for computing $\psi_{\widetilde{\mathcal{S}}}$ and its integral on ω. Finally, we discuss its application to robot path planning problem, when $N = 2$, and extend the method for computing least cost paths with homotopy class constraints using graph search techniques to high dimensional Euclidean configuration spaces. This in turn allows us to plan in the 4-dimensional configuration space with dynamic 3-dimensional obstacles ($X - Y - Z - Time$ space) with homotopy class constraints.

The notion of χ-homotopy has apparent connections with singular and cellular homology theories [35] and the De Raham cohomology [22] theory. However in our analysis we have mostly used the cobordism theory [24, 25] for development of the said differential form and the invariant. We are currently investigating the underlying connection of such an approach with the more standard homology and cohomology theories.

3.2 Metric information using graph search techniques – Voronoi Tessellation in Non-convex and non-Euclidean metric spaces Using Graph Search

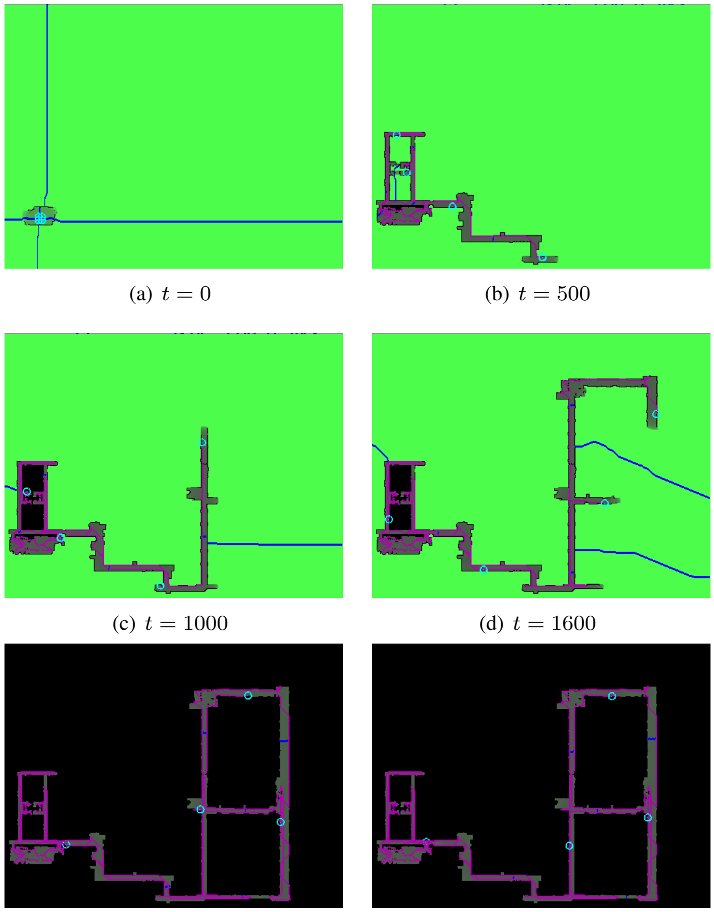
A graph is essentially a simplicial 1-complex whose dual describes a discretization of the configuration space. That is, each node of the graph is associated with a discretization cell (typically its centroid). This fact gives us a way of computing measures for the configuration space. In the simplest case one can compute a weighted volume of the configuration space by expanding nodes of the graph using a Dijkstra's kind of algorithm. While a more interesting application involves a wave-front like algorithm for partitioning the environment among multiple robots.

We apply this concept to multi-robot exploration and coverage in unknown non-convex environments [8] using an entropy-based metric to capture uncertainty in the environment. We use a search-based algorithm for computing a geodesic Voronoi tessellation in discrete environments. A *pseudo-centroid* of the Voronoi cells given a tessellation of a discrete environment, permits the application of centroid-based robot control laws for cooperative coverage and exploration in a distributed manner (Figure 6).

3.3 Metric Transformation for Efficient Optimal Planning in Environments with Nonuniform Metric

Graph-search based planning methods are well-suited for non-uniform metric spaces with holes/punctures. The only thing that a non-uniform metric space makes different for the graph laid down on the environment is that the edge costs become function of the location of the edges instead of just their lengths. However, in order to capture the non-Euclidean metric of the environment suitably, the discretization needs to be sufficiently fine. For example, one may use an uniform or unstructured fine discretization, but not a visibility graph. But this condition eventually fires back since the least cost path in the graph created by fine discretization of an environment is most often not the least cost path in the continuous metric space. For example, in Figure 1(a), the discretization scheme used was an uniform 8-connected grid-world (*i.e.* an uniform discretization into square cells, with the nodes of the graph placed at the centroid of each cell, and then each node connected to their 8 neighbors). As a result, segments of the trajectories were constrained to head in directions that are multiples of $45°$. Thus, the trajectories we obtain, although least cost paths in the graphs, are not the shortest in the original metric space. In a Euclidean metric setting one can use certain post-processing and smoothening methods to *straighten* the paths using notions of visibility. Likewise, in Euclidean metric one can employ a visibility graph, hence obtaining paths that are truly least cost, as demonstrated in Figure 1(b). But visibility graphs or post-processing and smoothening cannot be employed easily in non-Euclidean metric spaces with punctures.

Thus one question that we would like to answer: *Given a non-Euclidean metric space (with holes/punctures), is it possible to find a transformation that transforms the non-Euclidean metric to a Euclidean metric space? Or a metric space where we can exploit methods using visibility with ease?* If it can be done, we can set up a visibility graph or perform post-processing and smoothening in the transformed space, and/or plan the least cost trajectory in the transformed space. By the



(a) $t = 0$ (b) $t = 500$

(c) $t = 1000$ (d) $t = 1600$

(e) $t = 2600$ (complete map built) (f) $t = 2750$ (convergence)

Figure 5: Exploration and coverage of a large unknown environment. Green indicates uncertainty. The blue lines indicate the boundaries of the Voronoi tessellations of the 4 robots.

definition of transformation of metric space, the least cost path in the transformed space, when inverse-transformed back to the original space, will give the least cost path in the original space.

As we will discuss in a bit, one class of transformations that gives partial answer to this question are the *conformal mappings*. All metric spaces that transforms into a Euclidean space under some conformal mappings, or the metric spaces that are derived from Euclidean spaces using conformal mappings, can be conveniently transformed into Euclidean spaces. Such metric spaces are essentially the parabolic Riemann surfaces [31]. It is a well-known fact that holomorphic (complex analytic) functions define such conformal maps.

3.3.1 Problem definition in 2 dimensions with isotropic metric, and its partial solution

We will first consider the problem in a 2-dimensional isotropic metric space homeomorphic to $\mathbb{R}^2$. Let us denote a subset of this space (which is of interest to us) by $\mathbf{S}$, and assume it is equipped with the metric tensor $\mathbf{g}$ (which, by the assumption of isotropy, is a scalar multiple of identity). We are also provided with a chart, C, (*i.e.* coordinate system) such that a point in $\mathbf{S}$ is uniquely represented by the pair of values $(x, y) \in \mathbb{R}^2$. The metric and the chart are such that the matrix representation of $\mathbf{g}$ in the chart C is given by,

$$g(x,y) = \begin{bmatrix} m^2(x,y) & 0 \\ 0 & m^2(x,y) \end{bmatrix} = m^2(x,y) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

where the function m is given/known. Thus the length/cost of a differential element is given by $ds = m(x,y)\sqrt{dx^2 + dy^2}$. Thus, m can be interpreted as a cost map which acts as a scaling on the Euclidean length cost of a Euclidean metric.

The problem under consideration may now be posed as the problem of finding a chart $\overline{C}$ that is related to C in the following way,

$$\overline{x} = \overline{x}(x,y) \qquad , \qquad \overline{y} = \overline{y}(x,y) \tag{2}$$

such that the matrix representation of the metric tensor $\mathbf{g}$ in $\overline{C}$ is the identity matrix. That is,

$$\overline{g}_{kl} = \frac{\partial x_p}{\partial \overline{x}_k} \frac{\partial x_q}{\partial \overline{x}_l} g_{pq} = \delta_{kl} \tag{3}$$

where, in the second term in the above equation, summation is implied over repeated subscripts (*Einstein summation convention*) and the subscripts 1 and 2 are to indicate x and y respectively for notational convenience. δ_{kl} is the *Kronecker delta*.

It is a known fact [28] and not difficult to show that if we can find a holomorphic (complex analytic) function, f, such that $\|f(x+iy)\| = m(x,y)$ everywhere in $\mathbf{S}$, then the transformation defined by

$$\begin{aligned} \overline{x}(x,y) &= Im\left(\int f(z)dz\right)\Big|_{z=x+iy} + \kappa_0 \\ \overline{y}(x,y) &= \mp\, Re\left(\int f(z)dz\right)\Big|_{z=x+iy} + \kappa_0 \end{aligned} \tag{4}$$

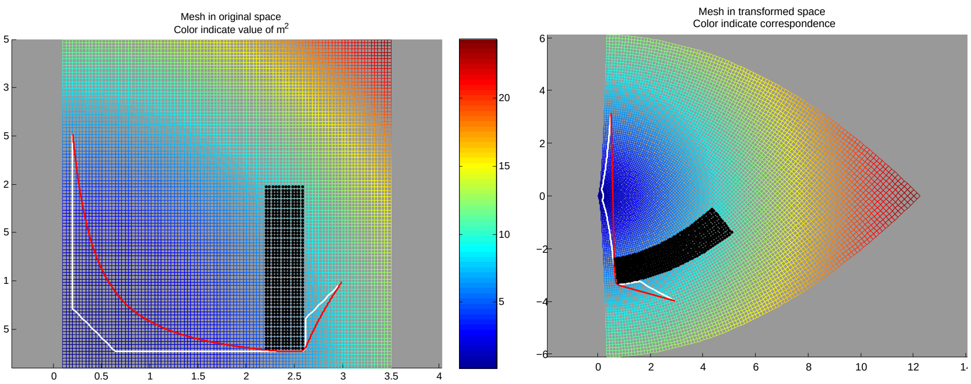
for some arbitrary integration constant κ_0, satisfies the condition in (3).

3.3.2 An Illustrative Example

In this section we present an illustrative example of the above discussion. We assume $m(x,y) = \sqrt{x^2+y^2}$ is the given cost map. Then we immediately obtain by observation, $f(z) = z$. Thus, $\overline{x}(x,y) = Im(z^2/2) = xy$ and $\overline{y}(x,y) = -Re(z^2/2) = (y^2 - x^2)/2$. We note that this map is invertable in the positive quadrant.

Figure 6(a) shows a mesh color coded with the value of m^2. The labels on the axes represent the values of x and y. The mesh is then transformed into the *barred* coordinate system to obtain the figure in 6(b). The color of the mesh is left same as the original one for ease of comparison.

An 8-connected grid-world is laid down in the original coordinates (Figure 6(a)) and a graph is hence formed. The white trajectory in Figure 6(a) demonstrate the least cost path for a given start and end coordinates in the 8-connected grid graph. As one can observe, the direction of the tangents on this path are restricted to multiples of $\pi/4$, which invariably leads to suboptimality. In an Euclidean setting one way of reducing the sub-optimality is to post-process the path by replacing portions of the path by line segments such that the segments do not intesect obstacles. However in a



(a) Mesh in original (*unbarred*) coordinates. Axes show values of (x, y). Color indicates the square of the cost (m^2).

(b) The same mesh in the transformed (*barred*) coordinates. Axes show values of $(\overline{x}, \overline{y})$. Colored to visualize the correspondence of points with the original mesh.

Figure 6: The black dots indicate obstacles. *Figure (a):* White path is the least cost path in the 8-connected grid graph planned in the *unbarred* coordinates. *Figure (b):* The same white path is shown in the *barred* coordinates. *Figure (b):* The red path in the *barred* coordinates is obtained by greedy post-processing of the white path. *Figure (a):* The red path in *unbarred* coordinates is obtained by inverse transformation.

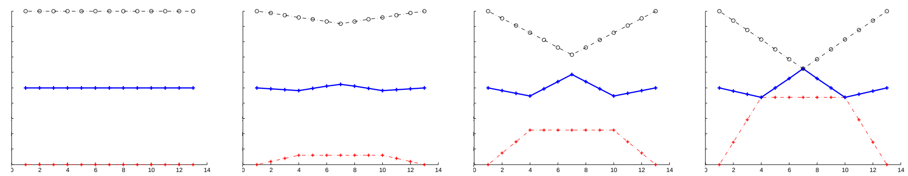
non-uniform metric like this, line segments are no more the geodesics. Hence this post-processing scheme is not possible in the *unbarred* coordinates.

However once we have transformed the white path to the *barred* coordinates (Figure 6(b)), the metric is Euclidean here. Hence we can now use the above-said post-processing technique in this coordinate. Thus, as indicated by the red path in Figure 6(b), we obtain a piece-wise straight segments in the *barred* coordinates, which is the least cost path. Performing an inverse transformation on this red path gived the corresponding least cost path in the *unbarred* coordinates (red path in Figure 6(a)).

3.3.3 Existence of f for a given m

Suppose we are given an arbitrary cost function, m, defined on $\mathbf{S}$. Can we always find a corresponding holomorphic f such that $\|f(x+iy)\| = m(x,y)$ on $\mathbf{S}$? As a first attempt one may try to numerically fit a complex analytic function to satisfy the given m. However, very quickly one can see that such an approach would result in vigorous oscillations in f, and in fact is not possible for general m.

One can see that on $\partial\mathbf{S}$, if we fix the real part of f, immediately from the Schwarz integral formula f gets determined over all of $\mathbf{S}$. From this it seems unlikely that one can find a f that satisfies $\|f(x+iy)\| = m(x,y)$ on the entirety of $\mathbf{S}$. Moreover, for the given metric, one can compute the scalar curvature of the space. If it is not identically zero, one cannot find an isometric embedding of the metric space in $\mathbb{R}^2$.



(a) Unconstrained plans, $k = 0$ (b) $k = 50$ (c) $k = 150$ (d) Converged solution, $k = 216$

Figure 7: Demonstration of convergence towards global optimal solution with progress of iterations. In each iteration only 1 robot updates its own trajectory. As we cycle through the robots, we increase the penalty for violating the rendezvous constraints.

3.4 Dimensional Decomposition – Distributed Optimization using Separable Optimal Flow

So far in our discussion we have encountered certain pros and cons of graph-search based path planning. However the biggest challenge that one encounters in discretization, graph creation and search problem is that the number of nodes, average degree and hence the complexity of the search algorithm increases exponentially with the dimensionality of the configuration space. The problem becomes evident in multi-robot planning problems where the configuration spaces of the individual robots cannot be decoupled due to presence of complex inter-robot constraints.

In this work [4] we try to exploit certain structures in the high dimensional configuration space and the constraints that prevent complete decoupling of the problem into lower dimensional planning problems, and hence use graph search techniques in concert with gradient ascent type of techniques in order to solve a class of constrained optimization problems. This particular class of optimization problem turns out to be well-suited for solving multi-robot path planning problem in cluttered non-convex environments with pair-wise constraints on their trajectories.

The intuitive concept behind the algorithm is that we start off by solving the global unconstrained problem, which is completely decoupled and hence can be solved as a bunch of lower dimensional problems. Then we gradually increase the penalty weights for violation of the constraints which are modeled as soft constraints, in a way not unlike dual and Lagrangian decomposition methods. We show that in every iteration of the algorithm, if we increase the penalty weights along certain specific directions (*Separable Optimal Flow Direction* and *Ascent Direction*) we are guaranteed to attain optimality and convergence in the limit (Figure 7). In order to deal with obstacles/punctures in the configuration space of individual robots, we need to do an exhaustive search in the different homotopy classes of trajectories. The tools developed in [5, 7, 6] can be used for that purpose.

4 Future Direction

4.1 Primary directions

4.1.1 Further investigation of χ-homotopy

As discussed earlier, for the present analysis we have used cobordism theory for defining the equivalence relation under consideration and designing the class invariants. We would like to re-formulate this equivalence in terms of more standard homology and co-homology theories.

We have introduced the notion of χ-homotopy for sub-manifolds of punctured Euclidean spaces. However there are definitive needs of extending this notion to more general manifolds. For example, the configuration space of a robotic arm with n joints is typically a n-torus. We hence plan to extend the theory to general punctured manifolds.

4.1.2 Investigation of mappings that can transform a non-uniform metric spaces into spaces where the notion of *visibility* can be exploited

As we have discussed, certain isotropic metric spaces let us use conformal mapping to transform them into Euclidean metric space. A Euclidean metric space allows us to use simple notions like visibility graph, or perform post-processing on trajectories found by graph-search algorithms on graphs created by fine discretization, in order to obtain trajectories that are optimal not only on the graph, but also in the metric space when inverse-transformed back to the original space.

Now we ask the question: *What is so special about the Euclidean metric that lets us use such convenient techniques?* The answer is essentially the ease of computation of the geodesic connecting two points. The notion of visibility is generalized to existence of geodesic segments between points in a punctured general metric space. More precisely, the determining factor for using visibility-based methods is the ease with which we can check whether the segment of a geodesic connecting points $\mathbf{p}_1$ and $\mathbf{p}_2$ intersect the segment connecting $\mathbf{p}_A$ and $\mathbf{p}_B$. Typically in an arbitrary metric space the computation of the geodesic passing through two given points is highly non-trivial. One can employ a method like *shooting method* for solving the Geodesic equation posed as a boundary value problem. However, in general, such methods are expensive and often practically infeasible.

Thus, our aim for this component of the proposal is two-fold:

i. Identification of metric spaces along with methods for each of them that allows us to check with considerable ease whether for a given set of points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_A$ and $\mathbf{p}_B$, the geodesic segment $\overline{\mathbf{p}_1\mathbf{p}_2}$ intersects the geodesic segment $\overline{\mathbf{p}_1\mathbf{p}_2}$.

ii. Identification of classes of metric spaces that can be transformed into one of the above mentioned special metric spaces, and the study of such transformations, for use of visibility-based methods.

In particular, we would like to investigate which metric spaces can be embedded in Euclidean spaces of same dimension (not necessarily isometrically) such that the geodesics are mapped injectively to straight lines in the Euclidean space (not necessarily isometrically). A non-trivial example being the hyperbolic plane and the *Beltrami-Klein model* of its embedding in $\mathbb{R}^2$. For such embeddings, the notion of visibility follows quite easily and naturally.

4.2 Other possible directions of investigation

4.2.1 Computation of *generalized centroid*

One of the challenging aspects of the problem in [8] was the computation of generalized centroid. Generalized centroid for a general non-convex region V is defined as,

$$\mathbf{C}_V^{gen} = \underset{\mathbf{p}\in V}{\operatorname{argmin}} \int_V f(d(\mathbf{q},\mathbf{p}))\phi(\mathbf{q})d\mathbf{q} \tag{5}$$

where ϕ is a weight function, and $d(\mathbf{q},\mathbf{p})$ is the shortest path (shortest Euclidean length) lying in V and connecting $\mathbf{q}$ and $\mathbf{p}$. For a convex region this reduces to the simple formula $\mathbf{C}_V =$

$\frac{\int_V \mathbf{q}\phi(\mathbf{q})d\mathbf{q}}{\int_V \phi(\mathbf{q})d\mathbf{q}}$. However for general non-convex regions this becomes infeasible to compute for real-time applications. In [8] we used certain workarounds. However we desire to investigate further into the problem to determine if there can be any better systematic way of computing generalized centroid, possibly approximately using graph-search based techniques.

4.2.2 Investigation of the condition for the existence of Separable Optimal Flow

Upon introducing the notion of "separable optimal flow" [4], we have shown that following separable optimal flow direction and ascent directions in an iterative fashion leads towards the global optimal. However we have not commented on or derived the conditions under which such directions can be guaranteed to exist. Thus, one important direction of future work will be to investigate such conditions if they exist. Another direction in which we would like to do some investigation is to see if we can generalize the constraints beyond the pair-wise restriction.

5 Conclusions

In this thesis we have described and proposed a rich collection of tools that attempts to create an amalgamation between graph-search based techniques and techniques from geometry and topology for solving robotics problems. We believe such studies and their further development will also help to bridge some of the still-existing gap between the active robotics community and the rich mathematics literature on subjects like algebraic geometry, algebraic topology, differential geometry and geometric topology.

6 Proposed outline of dissertation

Based on the completed and proposed work, the table below described the potential outline of the contents of the final thesis.

Proposed outline:

1. Introduction and literature review
2. Equivalence relation as constraints in planning problem
   i. Introduction
   ii. In 2 and 3 dimensions
   iii Generalization and extension to higher dimensions
   iv On non-Euclidean configuration manifolds
3. Voronoi tessellation and other measure-based techniques
   i. The search-based approach for non-convex regions
   ii. Application to coverage and exploration
   iii. On computation of generalized centroids of non-convex regions
4. Metric transformation for exploitation of visibility-based techniques
   i. Why visibility-based techniques?
   ii. The simplest case – conformal maps
   iii. The Beltrami-Klein model of the hyperbolic spaces, and the Real Projective space
   iv. Generalization
5. Dimensional decomposition of configuration space for efficient planning
   i. Motivation and challenges
   ii. Separable optimal flow and its properties
   iii. Application to distributed multi-robot path planning with pair-wise constraints
   iv. On existence of separable optimal flow directions
   v. Generalization of the constraints
6. Discussions and concluding remarks

Appendix

List of relevant refereed papers, and papers under review (in reverse chronological order):

A. Subhrajit Bhattacharya, Maxim Likhachev and Vijay Kumar (2011) *A Homotopy-like Class Invariant for Sub-manifolds of Punctured Euclidean Spaces* ***[Under Review]***. Discrete & Computational Geometry, Springer. arXiv:1103.2488 [math.DG]. (link)

B. Subhrajit Bhattacharya, Maxim Likhachev and Vijay Kumar (2011) *Identification and Representation of Homotopy Classes of Trajectories for Search-based Path Planning in 3D* ***[Accepted - To Appear]***. In Proceedings of Robotics: Science and Systems. 27-30 June. (link to draft PDF)

C. Subhrajit Bhattacharya, Hordur Heidarsson, Gaurav S. Sukhatme and Vijay Kumar (2011) *Cooperative Control of Autonomous Surface Vehicles for Oil Skimming and Cleanup* ***[Accepted - To Appear]***. In Proceedings of IEEE International Conference on Robotics and Automation (ICRA). 9-13 May. (link)

D. Subhrajit Bhattacharya, Nathan Michael and Vijay Kumar (2010) *Distributed Coverage and Exploration in Unknown Non-Convex Environments.* In Proceedings of 10th International Symposium on Distributed Autonomous Robotics Systems. 1-3 Nov, Springer. (link)

E. Subhrajit Bhattacharya, Vijay Kumar and Maxim Likhachev (2010) *Search-based Path Planning with Homotopy Class Constraints.* In Proceedings of The Third Annual Symposium on Combinatorial Search. Atlanta, Georgia, 8-10 July. (link)

F. Subhrajit Bhattacharya, Vijay Kumar and Maxim Likhachev (2010) *Search-based Path Planning with Homotopy Class Constraints.* In Proceedings of The Twenty-Fourth AAAI Conference on Artificial Intelligence. Atlanta, Georgia, 11-15 July. (link)

G. Subhrajit Bhattacharya, Vijay Kumar and Maxim Likhachev (2010) *Distributed Optimization with Pairwise Constraints and its Application to Multi-robot Path Planning.* In Proceedings of Robotics: Science and Systems. Zaragoza, Spain, 27-30 June, MIT Press. (link)

H. Subhrajit Bhattacharya, Maxim Likhachev and Vijay Kumar (2010) *Multi-agent Path Planning with Multiple Tasks and Distance Constraints.* In Proceedings of IEEE International Conference on Robotics and Automation (ICRA). Anchorage, Alaska, 3-8 May. (link)

I. Paul Vernaza, Maxim Likhachev, Subhrajit Bhattacharya, Sachin Chitta, Aleksandr Kushleyev and Daniel D. Lee (2009) *Search-based planning for a legged robot over rough terrain.* In Proceedings of IEEE International Conference on Robotics and Automation (ICRA). 12-17 May, pages 2380-2387. (link)

J. Subhrajit Bhattacharya, Sachin Chitta, Vijay Kumar and Daniel Lee (2008) *Optimization of a Planer Quadruped Dynamic Leap.* In Proceedings of 2008 ASME International Design Engineering Technical Conferences (IDETC). New York City, NY, 3-6 August. (link)

References

[1] Pramod Abichandani, Hande Y. Benson, and Moshe Kam. Multi-vehicle path coordination in support of communication. In *ICRA'09: Proceedings of the 2009 IEEE international conference on Robotics and Automation*, pages 3839–3846. Institute of Electrical and Electronics Engineers Inc., The, 2009.

[2] Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 2007.

[3] Dimitri P. Bertsekas and John N. Tsitsiklis. *Parallel and Distributed Computation:Numerical Methods*. Prentice Hall, 1989.

[4] Subhrajit Bhattacharya, Vijay Kumar, and Maxim Likhachev. Distributed optimization with pairwise constraints and its application to multi-robot path planning. In *Proceedings of Robotics: Science and Systems*, Zaragoza, Spain, 27-30 June 2010. [link].

[5] Subhrajit Bhattacharya, Vijay Kumar, and Maxim Likhachev. Search-based path planning with homotopy class constraints. In *Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence*, Atlanta, Georgia, July 11 2010. [link].

[6] Subhrajit Bhattacharya, Maxim Likhachev, and Vijay Kumar. A homotopy-like class invariant for sub-manifolds of punctured euclidean spaces. *Discrete & Computational Geometry*, 2011. [UNDER REVIEW] Electronic pre-print: arXiv:1103.2488 [math.DG].

[7] Subhrajit Bhattacharya, Maxim Likhachev, and Vijay Kumar. Identifying homotopy classes of trajectories for robot exploration and path planning. In *Robotics: Science and Systems*, 2011. [UNDER REVIEW].

[8] Subhrajit Bhattacharya, Nathan Michael, and Vijay Kumar. Distributed coverage and exploration in unknown non-convex environments. In *Proceedings of 10th International Symposium on Distributed Autonomous Robotics Systems*. Springer, 1-3 Nov 2010. [link].

[9] Frederic Bourgault, Alexei A. Makarenko, Stefan B. Williams, Ben Grocholsky, and Hugh F. Durrant-Whyte. Information based adaptive robotic exploration. In *in Proceedings IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS*, pages 540–545, 2002.

[10] W. Burgard, M. Moors, C. Stachniss, and F. Schneider. Coordinated multi-robot exploration. 21(3):376–378, June 2005.

[11] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. 20(2):243–255, April 2004.

[12] R. Delanghe, F. Sommen, and V. Souek. *Clifford Algebra and Spinor Valued Functions*. 1992.

[13] M Bernardine Dias, Robert Michael Zlot, Nidhi Kalra, and Anthony (Tony) Stentz. Market-based multirobot coordination: a survey and analysis. *Proc. of the IEEE*, 94(7):1257 – 1270, 2006.

[14] Brian Gerkey and Maja Mataric. A formal analysis and taxonomy of task allocation in multi-robot systems. *Int'l. J. of Robotics Research*, 23(9):939–954, 2004.

[15] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. Wiley-Interscience, 1994.

[16] D. Grigoriev and A. Slissenko. Polytime algorithm for the shortest path in a homotopy class amidst semi-algebraic obstacles in the plane. In *ISSAC '98: Proceedings of the 1998 international symposium on Symbolic and algebraic computation*, pages 17–24, New York, NY, USA, 1998. ACM.

[17] Eric A. Hansen and Rong Zhou. Anytime heuristic search. *Journal of Artificial Intelligence Research (JAIR)*, 28:267–297, 2007.

[18] John Hershberger and Jack Snoeyink. Computing minimum length paths of a given homotopy class. *Comput. Geom. Theory Appl*, 4:331–342, 1991.

[19] Peter Hilton. A brief, subjective history of homology and homotopy theory in this century. 61(5):282–291, December 1988.

[20] Nejat Karabakal and James C. Bean. A multiplier adjustment method for multiple shortest path problem. Technical report, The University of Michigan, June 1995.

[21] S. M. LaValle. *Planning Algorithms*. Cambridge University Press, Cambridge, U.K., 2006. Available at http://planning.cs.uiuc.edu/.

[22] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2002.

[23] M. Likhachev and D. Ferguson. Planning long dynamically-feasible maneuvers for autonomous vehicles. In *Proceedings of Robotics: Science and Systems (RSS)*, Cambridge, USA, June 2008.

[24] I. B. Madsen and R. J. Milgram. *The Classifying Spaces for Surgery and Cobordism of Manifolds*. Princeton Univ Pr.

[25] J. P. May. *A Concise Course in Algebraic Topology*. The University of Chicago Press, 1999.

[26] J. F. Nash. The imbedding problem for riemannian manifold. *Ann. of Math.*, 63:20–63, 1956.

[27] I. Necoara and J.A.K. Suykens. Interior-point lagrangian decomposition method for separable convex optimization. *Journal of Optimization Theory and Applications*, 143:567–588, 2009.

[28] Walter Rudin. *Real and complex analysis, 3rd ed.* McGraw-Hill, Inc., New York, NY, USA, 1987.

[29] Sikandar Samar, Stephen Boyd, and Dimitry Gorinevsky. Distributed estimation via dual decomposition. 2007.

[30] E. Schmitzberger, J.L. Bouchet, M. Dufaut, D. Wolf, and R. Husson. Capture of homotopy classes with probabilistic road map. In *International Conference on Intelligent Robots and Systems*, volume 3, pages 2317–2322, 2002.

[31] George Springer. *Introduction to Riemann Surfaces*. American Mathematical Society, January 2002.

[32] C. Stachniss. *Exploration and Mapping with Mobile Robots*. PhD thesis, University of Freiburg, Freiburg, Germany, April 2006.

[33] C. Stachniss, G. Grisetti, and W. Burgard. Information gain-based exploration using rao-blackwellized particle filters. pages 65–72, Cambridge, MA, June 2005.

[34] Sebastian Thrun, Wolfram Burgard, and Dieter Fox. *Probabilistic Robotics (Intelligent Robotics and Autonomous Agents)*. The MIT Press, 2005.

[35] James W. Vick. *Homology Theory: An Introduction to Algebraic Topology*. Springer, 1994.

[36] H. Zhang, V. Kumar, and J. Ostrowski. Motion planning under uncertainty. In *IEEE International Conference on Robotics and Automation*, Leuven, Belgium, May 16-21 1998. IEEE.