Time Independent and Time Dependent Catenary Problem

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Time Independent Catenary Problem

Preliminaries

Assumptions:

1. Gravity acts along negative Y axis.
2. Uniform mass per unit length.
3. The mass per unit length does not change with tension (*i.e.* non-elastic)

Notations:

1. $s =$ Length of the rope starting form an origin (determined by integration constants). *i.e.* Parametrization of the rope by its length.
2. $\mu =$ Mass per unit length of the rope (constant by assumption).

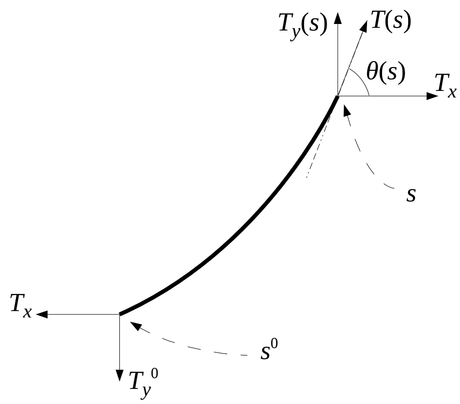

Figure 1: A segment of the stationary catenary.

3. $T(s) =$ Tension (magnitude of it) at s.
4. $T_y(s) = Y$ componet of tension at s.
5. $T_x = X$ componet of tension.
6. $\theta(s) =$ The angle made by the tangent or the tension vector with the positive X axis.

Lemmas:

1. It is a well-known result that tension in a rope/string, both static or dynamic, acts along the tangent of the rope/string at each point on it.
2. Since the only external force, gravity, acts along Y axis, the X component of the tension, T_x, at each point on the string will be uniform (can be verified by dwaring a FBD of an arbitrary segment of the rope).

Analysis

ODE relating x and y

From the FBD, $T_y(s) - T_y^0$ should be equal to the weight of the segment. Thus,

$$
\begin{aligned}
& T_y(s) = g\int_{s^0}^{s} \mu ds + T_y^0 \\
\Rightarrow \quad & T_x \tan(\theta(s)) = g\mu \int_{s^0}^{s} \mathrm{d}s + T_y^0 \\
\Rightarrow \quad & T_x \frac{dy}{dx} = g\mu \int_{x^0}^{x} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x + T_y^0 \\
\Rightarrow \quad & \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{g\mu}{T_x}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \qquad (1)
\end{aligned}
$$

Equation (1) is the static Catenary equation, and is a second order ODE. Solving it we get,

$$
y = \frac{T_x}{g\mu}\cosh\left(\frac{g\mu}{T_x}x + c_1\right) + c_2 \qquad (2)
$$

where c_1 and c_2 are integration constants detarmined by boundary conditions.

ODE in terms of s

We can write the Catenary equation in terms of the parameter s.

We get from equation (2),

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{g\mu}{T_x}x + c_1\right) \qquad (3)
$$

Using equation (3) we get,

$$
\begin{aligned}
s &= \int \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
&= \frac{T_x}{g\mu}\sinh\left(\frac{g\mu}{T_x}x + c_1\right) + s^0 \qquad (4)
\end{aligned}
$$

Thus using equation (4),

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}s} &= 1/\frac{\mathrm{d}s}{\mathrm{d}x} \\
&= \frac{1}{\cosh\left(\frac{g\mu}{T_x}x + c_1\right)} \\
&= \frac{1}{\sqrt{1+\sinh^2\left(\frac{g\mu}{T_x}x + c_1\right)}} \\
&= \frac{1}{\sqrt{1+\frac{g^2\mu^2}{T_x^2}(s-s^0)^2}} \qquad (5)
\end{aligned}
$$

Again, from equations (3) and (4),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g\mu}{T_x}(s - s^0) \tag{6}$$

Thus, using equations (5) and (6),

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\frac{g\mu}{T_x}(s - s^0)}{\sqrt{1 + \frac{g^2\mu^2}{T_x^2}(s - s^0)^2}} \tag{7}$$

Again, using equation(6),

$$\begin{aligned} T(s) &= \sqrt{T_x^2 + (T_y(s))^2} \\ &= T_x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\ &= T_x\sqrt{1 + \frac{g^2\mu^2}{T_x^2}(s - s^0)^2} \end{aligned} \tag{8}$$

Thus, from equations (5), (7) and (8), we obtain the Catenary equations in terms of the parameter s,

$$\left.\begin{aligned} &T(s)\frac{\mathrm{d}x}{\mathrm{d}s} - T_x = 0 \\ &T(s)\frac{\mathrm{d}y}{\mathrm{d}s} - g\mu(s - s^0) = 0 \\ &T(s) = T_x\sqrt{1 + \tfrac{g^2\mu^2}{T_x^2}(s - s^0)^2} \end{aligned}\right\} \tag{9}$$

Integrating equations (9) we obtain,

$$\left.\begin{aligned} x(s) &= \frac{T_x}{g\mu}\,\text{arcsinh}\left(\frac{g\mu}{T_x}(s - s^0)\right) + k_1 \\ y(s) &= \frac{T_x}{g\mu}\sqrt{1 + \frac{g^2\mu^2}{T_x^2}(s - s^0)^2} + k_2 \end{aligned}\right\} \tag{10}$$

where k_1 and k_2 are appropriate integration constants. One can easily eliminate s from equations (10) to obtain the equation (2).

Time Dependent Catenary Problem

Background

We start the analysis of the time-dependednt catenary problem by noting that the problem is equvalent to the problem of propagating wave through a string, but with additional consideration of gravity, innertia and large disturbances/ossilations. The standard wave equation used to solve the propagation of small disturbances through a string is given by [2],

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0 \tag{11}$$

However the derivation of this equation from the physics of the string assumes that the disturbances/ossillations in the string are small, the oscillations of points on the string take place in vertical direction only, the innertial forces are negligible compared to the tension forces, and that there is no gravity or external force acting on the string. In a time-varying catenary problem we cannot assume any of these.

In [1] a more generalized version of the partial differential equation describing propagation of disturbances through strings has been formulated that considers large ossilations of points on the string in both horizontal and vertical directions (both longitudinal and transverse waves simultaneously), existence of innertial forces as well as ellasticity of the material. However the formulation does not consider presence of external forces like gravity.

Our aim in the following sections will be to find a set of PDEs that represent propagation of large disturbances in a string in presence of gravity and innertial forces, but we will consider the material to be inellastic and the string to have uniform mass per unit length.

Preliminaries

Assumptions:

1. Gravity acts along negative Y axis (this can be assumed without loss of generality).
2. Uniform mass per unit length.
3. The mass per unit length does not change with tension (*i.e.* non-elastic)

Notations:

1. s = Parametrization of the string. *i.e.* A point at a distance s along the string from a chosen origin.
2. $(x(s,t), y(s,t))$ represents the position (coordinates) of a point on the string as a function of parameter s and time t.
3. μ = Mass per unit length of the string (constant by assumption).
4. $T(s,t)$ = Tension (magnitude of it) at s and time t.
5. $T_y(s,t) = Y$ componet of tension at s and time t.
6. $T_x(s,t) = X$ componet of tension at s and time t.
7. $\theta(s,t)$ = The angle made by the tangent or the tension vector with the positive X axis at s and time t.

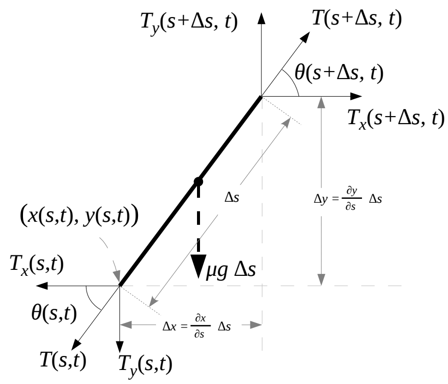

Figure 2: An infinidecimal element of the dynamic catenary at a particular instant of time.

Lemma:

1. It is a well-known result that tension in a rope/string, both static or dynamic, acts along the tangent of the rope/string at each point on it. This can be verified by doing a simple moment analysis of a differential element of the string.

Analysis

We write down the force equations of motion of a differential element of the string. (Note that from moment equation for a differential element all we obtain is that the tension acts along the tangent of the string [1].)

Since Δs is the length of the segment, $\mu \Delta s$ is its mass, and the net X and Y components of the tension acting upon it are $\frac{\partial T_x}{\partial s} \Delta s$ and $\frac{\partial T_y}{\partial s} \Delta s$ respectively. Thus, Newton's Second Law gives,

$$\mu \, \Delta s \frac{\partial^2 x}{\partial t^2} = \frac{\partial T_x}{\partial s} \Delta s \tag{12}$$

$$\mu \, \Delta s \frac{\partial^2 y}{\partial t^2} = \frac{\partial T_y}{\partial s} \Delta s - g\mu \, \Delta s \tag{13}$$

Now, by definition, $\Delta x = \frac{\partial x}{\partial s} \Delta s$ and $\Delta y = \frac{\partial y}{\partial s} \Delta s$ are the infinidecimal components of length of the segment Δs at time instant t. Thus from Figure 2 we can conclude, $\tan(\theta(s,t)) = \frac{\Delta y}{\Delta x}$ and $\Delta x^2 + \Delta y^2 = \Delta s^2$. The later relationship gives,

$$\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1 \tag{14}$$

But since T_x and T_y are the components of the tension at time t, which is alligned with the tangent at s, we have, $T_x(s,t) = T(s,t)\cos(\theta(s,t)) = T(s,t)\frac{\Delta x}{\Delta s}$ and $T_y(s,t) = T(s,t)\sin(\theta(s,t)) = T(s,t)\frac{\Delta y}{\Delta s}$. Thus we have,

$$T_x(s,t) = T(s,t)\frac{\partial x}{\partial s} \quad \Rightarrow \quad \frac{\partial T_x}{\partial s} = T\frac{\partial^2 x}{\partial s^2} + \frac{\partial T}{\partial s}\frac{\partial x}{\partial s} \tag{15}$$

$$T_y(s,t) = T(s,t)\frac{\partial y}{\partial s} \quad \Rightarrow \quad \frac{\partial T_y}{\partial s} = T\frac{\partial^2 y}{\partial s^2} + \frac{\partial T}{\partial s}\frac{\partial y}{\partial s} \tag{16}$$

Substituting equation (15) into (12), and (16) into (13), and from (14), we obtain the following set of equations that govern the motion of a catenary.

$$\left.\begin{aligned} &\frac{\partial^2 x}{\partial t^2} = \frac{1}{\mu}\left(T\frac{\partial^2 x}{\partial s^2} + \frac{\partial T}{\partial s}\frac{\partial x}{\partial s}\right) \\ &\frac{\partial^2 y}{\partial t^2} = \frac{1}{\mu}\left(T\frac{\partial^2 y}{\partial s^2} + \frac{\partial T}{\partial s}\frac{\partial y}{\partial s}\right) - g \\ &\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2 = 1 \end{aligned}\right\} \tag{17}$$

Thus we have three partial differential equations in three unknowns, namely, x, y and T, that describe the time-varying or dynamic catenary. With appropriate boundary conditions we may be able to solve these.

Also, it's worth noting that when x, y and T are independent of t, the first two equations in the set (17) reduce to,

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x}{\mathrm{d}s}\right) = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}s}\left(T\frac{\mathrm{d}x}{\mathrm{d}s}\right) = g\mu \end{aligned}\right\} \tag{18}$$

which upon integration and choosing the integration constants appropriately reduce to the first two equations of set (9), as expected. Moreover the third equation of the set (9) follows directly from the above two and $\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1$. Thus we find that the PDEs involving t that describe the dynamic catenary reduce to the ODEs describing the static catenary problem when the time dependence is removed.

References

[1] Subhrajit Bhattacharya. A study on a generalized wave equation for disturbances propagating through a one-dimensional medium placed in a two-dimensional space. In *Proceedings of International Conference on Theoretical, Applied, Computational and Experimental Mechanics*, pages 235–237, Kharagpur, India, 2004.

[2] Erwin Kreyszig. *Advanced Engineering Mathematics: Maple Computer Guide*. John Wiley & Sons, Inc., New York, NY, USA, 2000.